International Journal of Theoretical Physics, Vol. 23, No. 3, 1984

Third Quantization Formalism for Hamiltonian Cosmologies

N. Caderni

Institute of Astronomy, University of Cambridge, Cambridge, CB3 OHA, UK
and
Istituto Nazionale di Fisica Nucleare, Sezione di Pavia 27100 Pavia, Italy

M. Martellini

Istituto Nazionale di Fisica Nucleare, Sezione di Pavia, Pavia I 27100 Italy
and
Istituto di Fisica dell'Università, Milano 20133, Italy

Received February 24, 1983

Within the ADM technique of Hamiltonian cosmology, in the case of Bianchi class A models, we introduce a Fock-like, field theoretical approach to the description of quantum cosmology. We then calculate the total transition amplitude between two quantum states of the state functional of the ADM geometry.

1. INTRODUCTION

It is still an open problem whether the evolution of the very early stages of general relativistic cosmologies is influenced by quantum gravitational effects. The problem is, not only whether to believe in a drastic change of character of a particular cosmology, but also how to calculate in a reliable way these quantum gravitational effects. It is fair to say that at present we are far away from a satisfactory theory of quantum gravity and that we are pushed by necessity, in dealing with this matter, toward the use of approximate methods.

On the other hand we may look for the possible connections between some particular problems in gravitation and similar procedures and/or techniques successfully used in other nonlinear field theories, and it is in this spirit that we present our paper. Thus this work is not concerned with quantum gravity per se [for a review of the canonical formalism of quantization, usually used in quantum cosmology, see K. Kuchar (1981)], but rather with the stability under quantum effects of homogeneous cosmological solutions, and we try to address the question of how a particular cosmological configuration would evolve very close to the singularity.

One of the first attempts at the understanding of these issues is the investigation of the so-called quantum cosmological models, first proposed by Misner (1969) and later reviewed by MacCallum (1975). Quantum cosmological models considered so far in the literature rely on the ADM formulation of Hamiltonian cosmology. At first it was realized that it is possible to regard certain kinds of cosmological models as the solution of particular scattering problems, that is, to consider the motion of a point particle in a given potential. Later the usual procedure of Schrödinger–Klein–Gordon to quantize this system was applied. One of the main results of that investigation is that the universe wave packet, which replaces the universe point is nonquantized Hamiltonian mechanics, is not spread out in time and evolves toward a singularity as in the classical case.

In this paper, to achieve a quantum description of the homogeneous gravitational fields we propose a "third quantization" formalism of gravity. The paper starts with a theory in which the components of the metric are operators, i.e., a "second quantized" form, and then makes the state functions of that second quantized theory into operators, realizing in this way a "third quantization" of the gravitational field. The radical nature of this formalism is made clearer by noticing that the action being quantized is

$$S=\int d\Omega\, d\beta_+\, d\beta_- \left[(\partial_\mu\psi)^2(\beta_+,\beta_-,\Omega)+R(\beta_+,\beta_-,\Omega)\psi^2(\beta_+,\beta_-,\Omega)\right].$$

Here the "coordinates" Ω, β_+, β_- are those of the superspace, so that the most striking feature of this action is that the integral is over superspace and not over space-time like every other standard action.

Let us stress again that our main interest is, so to speak, more practical than formal, believing, as we do, that there is very little conventional wisdom one can appeal to in quantum gravity, and that we should not *a priori* close our doors to new methods. By the same token, we do not wish, in presenting an approach whose shortcomings will be clear from the beginning, to impose on our readers a blind faith in the truth of our conclusion, but only to stimulate them to investigate potentially powerful techniques.

In a quantum field theoretical approach, like the one described in this paper, we can consider the ADM potential as an interaction Lagrangian term and study the problem as a perturbative one. This way of treating the super-Hamiltonian was first advocated by Isham (1976) and, more recently, has been developed by Pilati (1981) into what is called "strong coupling quantum gravity." To be more precise, rather than study the evolution of a wave described by a Klein–Gordon type equation, we pass to a formalism of quantum field theory in which the Klein–Gordon field is an "operator field" (which we shall call the "universe field") and we calculate the transition amplitudes between two quantum states of the associated Fock space. The more natural interpretation of this formalism is that the Hilbert space consists of states containing an arbitrary number of universes and the creation and annihilation operators create and annihilate universes. We restrict ourselves to Bianchi–Behr models of type A^6 (namely, Bianchi types I, II, VI_0, VII_0, and IX) where it has been proved that Hamiltonian techniques are applicable. Then, using a perturbative Feynmann and Dyson-like S-matrix formalism on the superspace, we find the corresponding finite transition amplitudes.

Let us clarify here some aspects of our approach to the third quantization of the ADM cosmological models. We adopt an "external" point of view (Thirring, 1958) for the ADM potential; namely, we regard it as a fixed c-number function of superspace. The consequence of this interaction is that the Heisenberg field operator (represented by the universe field) creates virtual intermediate states which are excitations of the physical (asymptotic) states which represent quantum Kasner universes. This may be understood as a production of virtual universes which will have as a consequence that certain transition channels between initial and final universe states are forbidden at least in the framework of perturbative theory. This might be interpreted as due to the fact that nontrivial ADM perturbations have moved out of the initial quantum Kasner states of the asymptotic Fock space.

2. FIELD QUANTIZATION FORMALISM FOR HOMOGENEOUS CLASS-A COSMOLOGIES

In what follows we shall consider space-times whose metric can be written in the form (we use geometrical units $c = \hbar = 8\pi G = 1$)[6]

$$ds^2 = -dt^2 + R^2 e^{-2\Omega} e^{2\beta_{ij}} \omega^i \omega^j \tag{1}$$

where Ω is a monotonic function of the time t, R is an arbitrary scale factor, β_{ij} is a 3×3 symmetric time-dependent matrix, and ω^k are the time-independent differential one-forms obeying the relations $d\omega^i = \frac{1}{2} c^i_{jk} \omega^j \Lambda \omega^k$. Moreover, c^i_{jk} are the structure constants of the group of motions admitted by the space-like hypersurface of the metric (1). From our assumption on the monotonic time dependence of Ω, it follows that the ADM formalism promotes Ω as a time-labeling coordinate. The same formalism prescribes a canonical writing for the Einstein action I which, in the case of Bianchi class-A models, reads

$$I = \int P_+ \, d\beta_+ + P_- \, d\beta_- - H \, d\Omega \tag{2}$$

where H^2 is defined by the classical constraint

$$H^2 - P_+^2 - P_-^2 - e^{-4\Omega}[U(\beta_+, \beta_-) - 1] = 0 \tag{3}$$

Here $\beta_\pm$ are such that $\beta_{ij} = \text{diag}[\beta_+, (\sqrt{3}/2)\beta_- - (\beta_+/2), -(\sqrt{3}/2)\beta_- - (\beta_+/2)]$, and $P_\pm$, which are defined in Ref. 6, are independent variables. They can be regarded as a parametrization of the momenta π_{ij} conjugated to the metric g_{ij}. Let us notice that the momenta $P_\pm$ and the intrinsic time variable Ω are functionals of the three-geometry g_{ij}. The ADM Hamiltonian (3) is the same as for a particle "the universe point" moving in the two-dimensional β_+, β_- plane, under the influence of the time-dependent potential $e^{-4\Omega}[U(\beta_+, \beta_-) - 1]$. The values of the potential U are listed in Ref. 6. A major advantage of the ADM formulations consists then in the reduction of the equations that govern the evolution of the universe to a form totally similar to ordinary scattering problems. The first step toward quantization of the system (3) can be done in the Schrödinger–Klein–Gordon representation, where we make the usual prescription

$$P_\mp \to \hat{P}_\mp \equiv -i\frac{\partial}{\partial \beta_\mp}, \qquad H \to \hat{H} \equiv -i\frac{\partial}{\partial \Omega} \tag{4}$$

One can also define $\hat{P}_\mu \equiv (\hat{H}, \hat{P}_+, \hat{P}_-)$ and $Z^\mu \equiv (\Omega, \beta_+, \beta_-)$ with $\mu = 0, 1, 2$, so that (4) becomes

$$P_\mu \to \hat{P}_\mu \equiv -i\frac{\partial}{\partial Z^\mu} \tag{5}$$

Because of the operational nature of the representation (4) or (5), the classical constraint (3) becomes a weak relation:

$$\mathscr{H}_\perp \psi = 0 \tag{6}$$

where $\mathscr{H}_\perp$, the so-called super-Hamiltonian, is given by

$$\begin{cases} \mathscr{H}_\perp \equiv \mathscr{H}_0 - e^{-4\Omega}[U - 1], \\ \mathscr{H}_0 \equiv \hat{H}^2 - \hat{P}_+^2 - \hat{P}_-^2 \end{cases} \tag{7}$$

The dynamics of the full theory is generated by $\mathscr{H}_{\perp}$. Our goal is to study the quantum theory generated by $\mathscr{H}_0$, which is similar to a three-dimensional Klein–Gordon operator, and then to include as a perturbation the $e^{-4\Omega}[U-1]$ term as from equation (7). Then the free dynamics is controlled by

$$\mathscr{H}_0\psi_0 = 0 \tag{8}$$

where ψ_0 is a state functional which defines the domain of $\mathscr{H}_0$. Equation (8) determines the development of ψ_0 in the intrinsic time Ω and in the intrinsic space coordinates (β_+, β_-). Of course, ψ_0 is actually a functional of the metric g_{ij}, i.e., $\psi_0 = \psi_0[g_{ij}, \Omega]$. The fact that ψ_0 depends only on three degrees of freedom, and not on the six independent components of g_{ij}, is a direct consequence of the general covariance associated with changes of coordinates on a spacelike hypersurface. In a general case one would be left with a three-gauge freedom. In our case this is fixed by the requirement of spatial homogeneity. Roughly speaking, at this stage we are treating the quantum theory of the vacuum Bianchi geometry as a ficticious finite-dimensional quantum mechanics on the space of g_{ij} (the true geometry has not been quantized at all). This fact becomes more apparent if we rewrite equation (6), making use of (5), and we get

$$^3\Box\psi(\beta_+, \beta_-, \Omega) + e^{-4\Omega}\left[U(\beta_+, \beta_-) - 1\right]\psi(\beta_+, \beta_-, \Omega) = 0 \tag{9}$$

where $^3\Box \equiv \eta^{\mu\nu}(\partial/\partial z^{\mu})/(\partial/\partial z^{\nu})$ and $\eta^{\mu\nu} \equiv (+, -, -)$. Equation (9) is a three-dimensional, massive, Klein–Gordon equation with $e^{-4\Omega}[U-1]$, a Ω-time-dependent, masslike term. Its plane wave solution, i.e., the ψ_0, describes an independent quantum Kasner universe (i.e., Bianchi type-I solution). Because increasing (decreasing) Ω corresponds to an expanding (contracting) universe, we define each positive (negative) frequency state of ψ_0 as that describing an expanding (contracting) Kasner geometry. Since in this paper we assume the "spatial homogeneity," there is only one quantum Kasner solution to describe the whole universe.

In what follows we introduce a Fock space structure for the space of the states of $\hat{H}_0$. To this aim first we define a Hilbert space structure G on $\psi_0(\beta, \Omega)$. We start from the definition of the space of the square integrable functions $u(\beta_+, \beta_-)$ with the inner product:

$$\langle u|u'\rangle \equiv \int d\beta_+\, d\beta_-\,(u^*u') \tag{10}$$

The operator $^2\Delta \equiv -(\partial^2/\partial\beta^2 + \partial^2/\partial\beta^2)$ is positive definite and self-adjoint, with respect to (10) on this space. Then $^2\Delta$ possesses a complete, orthonormal set of eigenstates u_{E_p}

$$^{2}\Delta u_{E_p} = E_p^2 u_{E_p} \tag{11}$$

In order to define G we now have to include the Ω-time dependence into the states u_{E_p}. There seems no unique way of doing this. Following the suggestion of Ref. 8, we define the new state function $\psi_0(\beta_+, \beta_-, \Omega)$ as linear combination of the two Ω-dependent states $u_{E_p}\exp(iE_p\Omega)$ for any $E_p > 0$. We then give a (pre)Hilbert structure $\mathcal{G}$ to the space of $\psi_0(\beta_+, \beta_-, \Omega)$, if we endow it with the following inner product:

$$\begin{cases} \langle \psi_0(\Omega) | \psi_0'(\Omega) \rangle \equiv \dfrac{1}{2} \displaystyle\int_{\Omega = \text{const}} d\beta_+ \, d\beta_- \left(\psi_0^*(\Omega) \dfrac{\overleftrightarrow{\partial}}{\partial \Omega} J\psi_0(\Omega) \right) \\ A \dfrac{\overleftrightarrow{\partial}}{\partial \Omega} B \equiv A \dfrac{\partial}{\partial \Omega} B - B \dfrac{\partial}{\partial \Omega} A \end{cases} \tag{12}$$

Here $J\psi_0(\Omega)$ is defined as

$$J\psi_0 = i\psi_0^{(+)} + (-i)\psi_0^{(-)}$$

where we have first used (see Ref. 8) the Killing vector $i\ \partial/\partial\Omega$ to decompose ψ_0 into positive and negative frequency parts

$$\psi_0 = \psi_0^{(+)} + \psi_0^{(-)}$$

One can show that this inner product is Ω independent and gives a positive definite norm. Moreover, the Klein–Gordon-like operator $\hat{H}_0$ is self-adjoint with respect to (12). With the introduction of the potential $e^{-4\Omega}[U-1]$ the free ψ_0 (we recall that they represent Kasner solutions) become asymptotic states, and the interacting states ψ, as from equation (6), corresponding to Bianchi models with potential, can be understood as excitation or scattering centers of the asymptotic states. In order to construct a Fock space we need also the multiparticle states. To obtain these states the first step is to perform a Fourier decomposition of the free states ψ_0 in terms of coefficients A_p and A_p^+:

$$\psi_0(Z^\mu) = \int \frac{d^3p}{\left[2E_p(2\pi)^3\right]^{1/2}} \left[A_p \exp(-ip_\mu Z^\mu) + A_p^+ \exp(ip_\mu Z^\mu) \right] \tag{13}$$

Here ψ_0, being a functional of a real metric g_{ij}, is assumed to be a hermitian operator and A_p and A_p^+ are taken to be adjoints of each other, hermitian operators. The following commutation relations at equal time Ω on the momentum representation of the superspace configurations are valid:

$$[A_{\mathbf{p}}, A_{\mathbf{p}'}^+] = \delta^{(2)}(\mathbf{p}-\mathbf{p}') \tag{14a}$$

$$[A_{\mathbf{p}}, A_{\mathbf{p}'}] = [A_{\mathbf{p}}^+, A_{\mathbf{p}'}^+] = 0 \tag{14b}$$

A_p and A_p^+ are known as annihilation and creation operators, respectively. Having defined A and A^+ we can now construct a ground state $|0\rangle$ as a Fock vacuum state, that is,

$$A\ |0\rangle = 0$$

$$\langle 0|0\rangle = 1$$

Let us notice that the ground state $|0\rangle$ is a functional of the metric g_{ij} and represents, in our picture, an ADM state with no quantum Kasner universes. The multiparticle state space F_n, $n > 1$, is now defined as the Hilbert space of states found by taking the span of the action of all powers of A_p^+ on $|0\rangle$, as usual for a Fock representation. Finally the Fock space $\mathscr{F}$ is defined as

$$\mathscr{F} = \bigoplus_{n=0}^{\infty} \mathscr{F}_n$$

where $\mathscr{F}_0$ is the complex number space and $\mathscr{F}_1$ is isomorphic to $\mathscr{G}$ previously defined.

At this juncture let us make a comment on the energy $E_p \cdot E_p$ was defined as the eigenvalue of the operator ${}^2\Delta$ in equation (11). This operator is self-adjoint on the space $\mathscr{G} \simeq \mathscr{F}_1$, and positive definite. The former property implies by the spectral theorem that E_p is a real quantity and can be used to label states since the operator ${}^2\Delta$ admits on $\mathscr{G}$ a complete set of eigenvectors. In the momentum representation (4) E_p^2, from equation (11), is given by

$$E_p^2 = \hat{P}_+^2 + \hat{P}_-^2 \tag{15}$$

and does not depend on $\hat{H}$, therefore is conserved in Ω time. For these properties, since E_p is (i) real, (ii) the eigenvalue of a complete set of quantum states (U_{E_p}), and (iii) conserved on the coordinate manifold $(\beta_+, \beta_-, \Omega)$, we are allowed to regard E_p as the "energy" of the free states

ψ_0. E_p is a measure of the expansion rate of Kasner solutions; however, it has nothing to do with what is normally called energy for a gravitational field.

We now have to take into account the effect of the interaction $e^{-4\Omega}[U-1]$ on the asymptotic states spanning the Fock's space $\mathscr{F}$. To this purpose we shall define, in the next section, an S-matrix operator in the Feynmann–Dyson formalism. In this context, it is more convenient to work with a Lagrangian formalism. We start by introducing an expansion parameter λ in the expression for the ADM potential, in such a way that the physical potential is taken when $\lambda = 1$. Thus a Lagrangian which can generate by variational principles equation (9) is given by

$$\mathscr{L}(Z;\lambda) \equiv \mathscr{L}_0(Z) + \mathscr{L}_1(Z;\lambda) \tag{16a}$$

$$\mathscr{L}_0(Z) \equiv -\tfrac{1}{2}\psi(Z)^3\Box\psi(Z) \tag{16b}$$

$$\mathscr{L}_1(Z;\lambda) \equiv -\tfrac{1}{2}\lambda V(Z;\lambda)\psi^2(Z) \tag{16c}$$

$$\mathscr{V}(Z;\lambda) \equiv \frac{1}{\lambda} e^{-4\Omega}[U(\mathbf{Z})-1] \tag{16d}$$

Equations (16) suggest that we regard the quadratic (in ψ) Lagrangian L_0 as the free Lagrangian, and L_1 as the interaction Lagrangian, to be treated, in a perturbative way, by the known S-matrix techniques. To be more precise we take L_1 as a c-number interaction, so that we shall work in an "external approximation" to the complete quantum field theory associated with the operator field ψ. The meaning of the Feynmann–Dyson formula for the scattering matrix developed in the next section is one of regarding higher powers of the Heisenberg operator ψ in S (corresponding to interactions between the field quanta themselves) as high quantum corrections in $\hbar$ to the free dynamics generated by the Kasner-like ADM states. We notice also that the Fock quantization of a Lagrangian like the one defined by (16) leads to a renormalizable theory.

3. COMPUTATION OF THE TRANSITION AMPLITUDES

The Feynmann–Dyson formula for the S matrix on the superspace reads (Lurie, 1968)

$$S = T\exp\left(i\int d^3Z\left\{-\frac{\lambda}{2}V(Z,\lambda)N[\psi^2(Z)]\right\}\right)$$

where T and N stand for time and normal ordered products, respectively.

Let us notice that the ADM potential is not treated as an infinite potential well but it is always taken of the form (16d). Therefore, similarly to the problems of interacting Lagrangian (in external approximation) which are bilinear in the fields, one can take that the perturbative series are asymptotic series. This means that the main contributions to the perturbative series come from low-order terms, so that we can restrict ourselves to the second-order contribution in the expansion parameter λ. We get that, up to terms of order λ^2, the scattering matrix is given by

$$
\begin{aligned}
S &= 1 + S^{(1)} + S^{(2)} \\
S^{(1)} &= -\frac{i\lambda}{2} \int d^3 Z_1 V(Z_1, \lambda) T\{ N[\psi^2(Z_1)] \} \\
S^{(2)} &= -\frac{\lambda^2}{8} \iint d^3 Z_1 d^3 Z_2 V(Z_1, \lambda) V(Z_2, \lambda) \times \\
& \quad T\{ N[\psi^2(Z_1)] \times N[\psi^2(Z_2)] \}
\end{aligned} \tag{17}
$$

In writing equation (17) we treated, as usual, $\mathscr{L}_1(Z, \lambda)$ in normal ordered form, where the ADM potential was taken to be an external interaction (c number). It is now possible to decompose the chronological products via the Wick theorem. This theorem gives

$$
\begin{aligned}
T\{ N[\psi^2(Z_1)] \} &= N[\psi^2(Z_1)] \\
T\{ N[\psi^2(Z_1)] N[\psi^2(Z_2)] \} &= N[\psi^2(Z_1) \cdot \psi^2(Z_2)] \\
& \quad - 4i\Delta_F(Z_1 - Z_2) N[\psi(Z_1) \cdot \psi(Z_2)] \\
& \quad + \text{(second-order vacuum bubble diagram contributions)}
\end{aligned}
$$

where Δ_F is the massless scalar Feynman propagator, defined by

$$
({}^3\Box_{Z_1} - i\varepsilon) \Delta_F(Z_1 - Z_2) = -\delta^{(3)}(Z_1 - Z_2)
$$

Moreover we observe that the badly divergent term due to the sum of all the possible bubble diagrams, corresponding to the vacuum-to-vacuum transition amplitudes $\langle 0|S|0\rangle$, appears as an overall multiplicative factor, i.e., $S = \langle 0|S'|0\rangle S_c$, where S_c is the connected scattering matrix. Since it is possible to prove that $\langle 0|S'|0\rangle$ is a trivial phase factor, we shall in the future omit all disconnected graphs and work with the connected part of the scattering matrix only. Thus we can write

$$S_c = 1 - \frac{i\lambda}{2}\int d^3Z_1 V(Z_1,\lambda) N[\psi^2(Z_1)]$$
$$-\frac{\lambda^2}{8}\iint d^3Z_1 d^3Z_2 V(Z_1,\lambda)V(Z_2,\lambda) N[\psi^2(Z_1)\psi^2(Z_2)]$$
$$+\frac{i\lambda^2}{2}\iint d^3Z_1 d^3Z_2 V(Z_1,\lambda)\Delta_F(Z_1 - Z_2)V(Z_2,\lambda)$$
$$\times N[\psi(Z_1)\cdot\psi(Z_2)] \tag{18}$$

We can now calculate the S_c-matrix elements between the "physical" states $|p,q\rangle$ and $|p',q'\rangle$. To this purpose we use the following formulas:

$$\langle q', p'|N[\psi^2(Z_1)]|p,q\rangle = \frac{1}{(E_{q'}E_q)^{1/2}}\exp[i(q-q')Z_1]$$

$$\langle q', p'|N[\psi(Z_1)\psi(Z_2)]|p,q\rangle = \frac{1}{2(E_{q'}E_q)^{1/2}}$$
$$\times[\exp(iqZ_2 - iq'Z_1) + \exp(iqZ_1 - iq'Z_2)]$$

$$\langle q', p'|N[\psi^2(Z_1)\psi^2(Z_2)]|p,q\rangle = \frac{1}{4(E_{q'}E_{p'}E_pE_q)^{1/2}}$$
$$\times\{4\exp[i(p-q')Z_1 + i(q-p')Z_2]$$
$$+\exp[-i(q'+p')Z_2 + i(p+q)Z_1]$$
$$+\exp[-i(q'+p')Z_1 + i(p+q)Z_2]\}$$

and we define $\hat{V}(k,\lambda)$, θ and $\hat{\Delta}_F(k)$ according to

$$V(Z,\lambda) = \frac{1}{(2\pi)^3}\int d^3k e^{ikZ}\hat{V}(k,\lambda)$$

$$\theta \equiv \mathrm{sign}(\hat{V})$$

$$\Delta_F(Z_1 - Z_2) = -\frac{i}{(2\pi)^3}\int d^3k \exp[ik(Z_1 - Z_2)]\hat{\Delta}_F(k),$$

$$\hat{\Delta}_F(k) = \frac{1}{k^2 - i\varepsilon}$$

It then follows that

$$\begin{aligned}
\langle q', p'|S_c|p, q\rangle = & 1 - \frac{i(\lambda/2)}{(E_{q'}E_q)^{1/2}}\hat{V}(q-p',\lambda) - \frac{\lambda^2/32}{(E_{q'}E_{p'}E_pE_q)^{1/2}} \\
& \cdot\left[\hat{V}(p-q',\lambda)\hat{V}(q-p',\lambda)\right. \\
& \left. + 2\hat{V}(p+q,\lambda)\hat{V}(-q'+p',\lambda)\right] - \frac{2\pi^3\lambda^2}{(E_{q'}E_q)^{1/2}} \\
& \cdot\int d^3k\hat{\Delta}_F(k)\left[\hat{V}(q'-k,\lambda)\hat{V}(k-q,\lambda)\right. \\
& \left. + \hat{V}(-k-q',\lambda)\hat{V}(k+q',\lambda)\right] \\
\simeq & 1 - \frac{1}{(E_{q'}E_q)^{1/2}}\left[\frac{i\lambda}{2}\hat{V}(q-q',\lambda)\right. \\
& \left. + 4\theta\pi^3\lambda^2\int d^3k(k^2-i\varepsilon)^{-1}\hat{V}^2(k,\lambda)\right] \\
& - \frac{1}{(E_{q'}E_{p'}E_pE_q)^{1/2}}\left[\frac{\lambda^2}{32}\hat{V}(p-q',\lambda)\hat{V}(q-p',\lambda)\right. \\
& \left. + \frac{\theta\lambda^2}{16}\hat{V}(p+q,\lambda)\hat{V}(p'+q',\lambda)\right]
\end{aligned} \tag{19}$$

The probability amplitude of a transition from an initial state $|p,q\rangle$ to a final state $|p',q'\rangle \neq |p,q\rangle$ is given by $\langle q', p'|R|p,q\rangle$, where the R matrix is defined by $S_c = I + iR$. Therefore from equation (19) we obtain

$$\begin{aligned}
\langle q', p'|R|p, q\rangle \simeq & -(E_{q'}E_q)^{-1/2}\left[\frac{i}{2}\hat{V}(q-q',\lambda)\lambda\right. \\
& \left. + 4\theta\pi^3\lambda^2\int d^3k\, k^{-2}\hat{V}^2(k,\lambda)\right] \\
& + O\left[(E_{q'}E_{p'}E_pE_q)^{-1/2}\right]
\end{aligned} \tag{20}$$

To proceed further along our calculations and compute the integral appearing in equation (20) we observe the following point: in the ADM formalism the evolution of the universe point, for a given homogeneous mode, is governed by the fact that the potential barrier is moving, rather than by the actual change in shape of the potential walls themselves as time goes by.

Therefore in calculating the Fourier transform of the potential V, it is sufficient to restrict oneself to the temporal-dependent part, whereas the purely spatial (β-dependent) part only contributes a form factor F on the momentum space. In order to compute the Fourier transform of V, the next point is to make the simultaneous analytic continuation $\Omega \to i\tau$ and $k_0 \to -ik'_0(\mathbf{k} \to \mathbf{k}')$, which leaves unchanged the representation (5). We then find in the linearized regime [at the order $O(\Omega)$]

$$\hat{V}(k', \lambda) = \hat{V}(k'_0, \lambda) F(\mathbf{k}')$$

$$\hat{V}(k'_0, \lambda) = \int d\Omega \left(\frac{1}{\lambda} e^{-4\Omega} \right) e^{-i\Omega k_0} \Big]_{\substack{k_0 \to ik'_0 \\ \Omega \to i\tau}} \simeq \frac{i\pi}{2\lambda} e^{-k'_0/4} \tag{21}$$

where we made use of the following formulas:

$$\exp(-x) \sim (1+x)^{-1}$$

and

$$\int \frac{dx}{2\pi} \left[(a+ix)^{-\nu} \right] e^{-ixy} = \begin{cases} [\Gamma(\nu)]^{-1} y^{\nu-1} e^{-ay}, & y > 0 \\ 0, & y < 0 \end{cases}$$

Here $\Gamma(\nu)$ is the γ function of (ν) and $F(\mathbf{k}') = \int d^2\mathbf{z}[U(\mathbf{z}) - 1] e^{i\mathbf{k}' \cdot \mathbf{z}}$.

We can now calculate the momentum integral of equation (20), after insertion of the expression (21). This integral is, however, ultraviolet divergent, while the infrared divergence can be avoided by staying away from vanishing momenta. To evaluate it, it will be sufficient to introduce a high-energy cutoff Λ_0, $\Lambda_0 \to \infty$, and to remember that

$$\int^u dx \left(\frac{e^{-x}}{x^2} \right) = -\frac{1}{u} e^{-u} - \log(u) e^{-u} + C$$

where C is the Euler constant ($C = 0.57721$). The momentum integral of equation (20) is then given, in the limit $\Lambda_0 \to \infty$, by

$$\begin{aligned} &-4\theta\pi^3\lambda^2 \int d^3k\, d^{-2}\hat{V}^2(k, \lambda) \\ &\simeq + i\theta\pi^5\lambda^2 A_{(0)} \int^{\Lambda_0} dk'_0\, k_0'^{-2} e^{-k'_0/2} + B(\Lambda_0) \\ &= -\frac{i}{2}\theta\pi^5 A_{(0)} f(\Lambda_0) \left[1 - A_{(0)}^{-1} F^{-1}(\Lambda_0) \right. \\ &\qquad \left. + g(\Lambda_0) A_{(0)}^{-1} f^{-1}(\Lambda_0) \right] + B(\Lambda_0) \end{aligned} \tag{22}$$

where we pose

$$f(\Lambda_0) = \log(\Lambda_0/2)\exp(-\Lambda_0/2)$$

$$g(\Lambda_0) = (\Lambda_0/2)^{-1}\exp(-\Lambda_0/2)$$

$$A(\Sigma) = \int_\Sigma d^2\mathbf{k}' F^2(\mathbf{k}') \equiv A_{(0)}$$

$$B(\Lambda_0;\Sigma) = i\theta\pi^5 \sum_{n\geqslant 1} (-1)^n \int^{\Lambda_0} dk_0' k_0'^{-2(n+1)} e^{-k_0'/2} \times \int_\Sigma d^2\mathbf{k}'\mathbf{k}^{2n} F^2(\mathbf{k}') \equiv B(\Lambda_0)$$

$$B(\Lambda_0) \propto \sum_{n\geqslant 1} (-1)^n A_{(n)}(\Sigma)\left[\int^{\Lambda_0} dk_0' k_0^{-2} e^{-k_0'/2}\right]$$

$$A_{(n)}(\Sigma) = \int_\Sigma d^2\mathbf{k}'\mathbf{k}'^{2n} F^2(\mathbf{k}') \equiv A_{(n)} \tag{23}$$

It is now worthwhile to make a few points clear. The potential V contains an expansion parameter λ. This parameter, as has already been remarked in Section 2, has been introduced in order to apply perturbative techniques, and does not represent a coupling constant of the model. The "physical" potential is $V(Z,\lambda)|_{\lambda=1}$ so that at the end of any perturbative calculations one has to make, the limit $\lambda \to 1$. Thus the "physical" transition amplitudes will be the ones with $\lambda = 1$. The integral (22) up to the B-term diverges as $A_0 f(\Lambda_0)$ for $\Lambda_0 \to \infty$, and multiplicative renormalization is then needed. Following the usual field-renormalization procedure, we rescale the wave function $\psi(Z)$ according to $\psi(Z) = W^{1/2}\psi_R(Z)$, where the subscript R means renormalized. The bare Feynmann propagator will then be written as a function of the renormalized one: $\hat{\Delta}_F(k) = W\hat{\Delta}_{F,R}(k)$, where $\hat{\Delta}_{F,R}(k) \sim 1/(k^2 - i\varepsilon)$ when $|k^2| \to 0$. The singular renormalization constant W is then fixed imposing the finiteness of the momentum integral (22), when replacing $\hat{\Delta}_F$ by $\hat{\Delta}_{F,R}$. We obtain that if

$$W^{-1} \sim \left[A_{(0)} + \sum_{n\geqslant 1} (-1)^n A_{(n)}\right] f(\Lambda_0) \tag{24}$$

as $\Lambda_0 \to \infty$, and the integral (22) becomes finite. An interesting circumstance is represented by the fact that, in fixing the renormalization constant

W, we remove the classical arbitrariness of the cosmic scale factor R, which is present in the metric (1) and in the ADM potential. In fact the ADM potential is always defined up to a scale factor R which is included for convenience in choosing the units. The physical interaction Lagrangian, that is to say the one expressed in terms of the renormalized quantities, will be written as

$$\mathscr{L}_{I\,\mathrm{phys}}(Z) = -\tfrac{1}{2} WR^2 V(Z,1)\psi_{\mathrm{phys}}(Z) \tag{25}$$

so that the renormalization procedure shifts the scale factor from R^2 to WR^2 and compels us to regard WR^2 as a coupling parameter.

Neglecting the terms of order $(E_{q'}E_{p'}E_q E_p)^{1/2}$, the renormalized "physical" reaction amplitude between the states $|q'\rangle$ and $|q\rangle$ is therefore given by (for $q' \neq q$)

$$R_{q'q} \equiv \langle q'|R|q\rangle_{\mathrm{Ren}} \simeq -\frac{i}{2}\left(E_{q'}E_q\right)^{-1/2} \times \hat{V}(q'-q,1) + (\text{finite parts}) \tag{26}$$

so that, by means of equation (21), the probability of transition $P_{qq'}$ takes the form [up to the form factor $F^2(\mathbf{q}'-\mathbf{q})$]

$$P_{q'q} \equiv |R_{q'q}|^2 \propto \frac{1}{E_{q'}E_q} e^{-\Delta E} \tag{27}$$

where $\Delta E \equiv |E_q - E_{q'}|/2$.

The transition amplitudes between given physical states describe the phenomenon through asymptotic states. In the case investigated here the asymptotic states are represented by an infinite ensemble of independent quantum Kasner universes. The intermediate states that mediate these asymptotic states can then be interpreted as scattering centers or as excitations of the quantum Kasner universes and are associated to vacuum Bianchi models with nontrivial ADM potential. One sees from equation (27) that the reaction probability decreases as $E_{q'}$ increases, *and one would conclude that the transition to outgoing large $E_{q'}$ state is very unlikely by quantum effects*. This means that the evolution is stable only for ADM asymptotic states characterized by low values of the "energy" E_p. The quantum fluctuations present in the interaction zone can completely destroy those Kasner configurations which are infinite in E_P. This phenomenon could be analogous to vacuum tunneling of instantons, with jumps between nonhomotopically connectable states. From the point of view of cosmology this picture shows the "unpredictability" of the evolution in the superspace between infinite energy Kasner configurations.

This third quantization formalism of gravity faces two basic difficulties: (i) the breakdown of causality. In fact, we are left with the simultaneous presence at the same time Ω of several physical universes all equally probable from the quantum mechanics point of view; (ii) the existence of (unphysical) transitions between multiuniverse states having different occupation numbers. Of course, these do not occur if one requires the conservation (in the super-space-time) of the occupation number $\Sigma_{\mathbf{p}} A_{\mathbf{p}}^{+} A_{\mathbf{p}}$.

Let us conclude by noticing that, in principle, this program can be applied to more complicated, inhomogeneous geometries, provided that the space-time has topology $\Sigma \times \mathbb{R}$, where Σ is a three-dimensional manifold, and supports an ADM formulation. In this case in addition to equation (6), one would have another weak relation which would be associated with the classical generator of the diffeomorphisms on Σ. This reflects the gauge invariance of the gravitational field. In the case of homogeneous geometries this gauge freedom is automatically frozen, owing to the existence of a natural group of isometries, which is the one furnished by the Bianchi group acting on Σ.

4. CONCLUSIONS

To answer the question of what the quantum fluctuations in a certain gravitational problem can be, we have reduced the corresponding field theory to the quantum mechanics ("the model") of a certain functional of the effective metric g_{ij} which is naturally given by the Hamiltonian formulation of cosmology. With this procedure we have by-passed the tremendous problems connected to the ultraviolet divergencies of the metric operator of quantum gravity. The fact is that the model here presented can, at most, present renormalizable self-energy (mass) insertions. However, as it is known, the renormalization procedures by subtraction of the ultraviolet divergencies have a physical meaning in that they allow to define the bare charges, masses, and so on which are present in a Lagrangian formulations as the observable quantities. This last aspect is missing in our approach, as well as in recent similar ones (see, e.g., Ref. 5). However, in quantum theories, there is another aspect besides the one of definibility of the quantum observables. This is the problem of defining a unitary evolution of quantum states. In a now old literature, this was the point of view taken, for instance, by the semiphenomenological, S-matrix, approaches to the models of strong interactions between elementary particles, which are not formulable in the framework of the usual perturbative theory of quantum fields. Naturally, our third quantization procedure is motivated for a nonrenormalizable theory, like Einstein theory of gravity, where it is not yet possible to give, at the same time, a description of quantum states together with their definibility as states forming renormalizable expectation values. We can only then privilege one aspect of the quantum formulation of the gravitational field. With the present work we developed a model and a formalism that allows us to describe the quantum evolution of a particular type of gravitational field, that is, the one described by vacuum geometries of Bianchi type A.

From our point of view, the stronger suggestion of this work is the possibility of treating certain vacuum gravitational solutions as excited states of Kasner solutions. In this sense we can note a certain analogy with nonlinear field theories that present solitonic solutions as collective excitations of quanta. Maybe the problem of quantizing the gravitational field could be approached in an analogous fashion, quantizing only inside classes of solutions of the Einstein equations (Martellini, 1979).

The program proposed in this paper is essentially a third quantization of gravity. As such it "would represent a radical departure from the conventional wisdom of ordinary quantum field theory" (Kuchar, 1981) and has therefore never been seriously considered by physicists. In view of the lack of success to data of any other program for quantizing gravity, this one should not be rejected out hand. For instance, the application of this program to "scalar QED" in the temporal gauge ($A_0^+|0\rangle = 0$) gives a wave equation for the state functional $\psi(\beta, \Omega)$ of the form

$$ {}^2\Box\psi(\beta,\Omega) - V(\beta)\psi(\beta,\Omega) = 0 \tag{28} $$

$$ {}^2\Box \equiv \frac{\partial^2}{\partial\Omega^2} + \frac{\partial^2}{\partial\beta^*\partial\beta} $$

$$ V(\beta) \equiv \beta^*\left(m^2 + p_i^2 + ieA_i p_i + e^2 A_i^2\right)\beta $$

Here the "superspace-coordinates" β^* and β are related, respectively, to the complex boson fields ϕ^* and ϕ. Moreover, the c numbers m and p_i (they stand, respectively, for the mass and the spatial momenta for the free bosons of the "first and second quantization" of the theory) must be understood as arbitrary parameters for the functional $\psi(\beta, \Omega)$. Having assumed (β^*, β) as operators, equation (28) formally describes a "second quantization" of the scalar QED. Then, the "third quantization" formalism previously developed for the gravitational field starts when one makes the state functional of (28) an operator-valued distribution. We think that

$\psi(\beta, \Omega)$ is related in a same way to the tree approximation of the path integral of the scalar QED (in the temporal gauge). Therefore the S-matrix technique discussed in the above sections for the "Heisenberg field operator" associated with ψ, is only a perturbative trick which reflects in this contest the ordinary loop expansion around a saddle point of the path integral. However, many questions remain to be asked and answered, in particular with respect to the application of this formalism to the standard quantum field theory, and the work sketched above can only be regarded as the germ of a new research program on the "quantum geometry," which we think is far from being dead.

ACKNOWLEDGMENTS

One of the authors, M. M., would like to thank K. Kuchar for helpful discussions. The research was supported in part through National Science Foundation Grant No. PHY77-27084.

NOTE ADDED IN PROOF

After the submission of this paper we received a preprint by J. B. Hartle and S. W. Hawking (Wave Function of the Universe, Santa Barbara preprint, 1983) where a problem similar to ours is discussed.

REFERENCES

Ashtekar, A., and Magon, A. (1975). *Proceedings of the Royal Society of London Series A*, **346**, 375.

Isham, C. J. (1976). *Proceedings of the Royal Society of London Series A*, **351**, 209.

Kuchar, K. (1981). In *Quantum Gravity 2*, C. J. Isham, R. Penrose, and D. W. Sciama, eds. Clarendon Press, Oxford, with J. B. Hartle, in "Contribution to the Bryce DeWitt 60th Birthday Festschrift," to appear and references therein quoted.

Lurie, D. (1968). *Particles and Fields*. Interscience Publishers, New York.

MacCallum, M. A. H. (1975). In *Quantum Gravity, an Oxford Symposium*, C. J. Isham, R. Penrose, and D. W. Sciama, eds. Clarendon Press, Oxford.

Martellini, M. (1979). *Nuovo Cimento*, **54B**, 21.

Misner, G. W. (1969). *Physical Review*, **186**, 1319; also in *Magic Without Magic*, J. R. Klander, ed., p. 441. Freeman, San Francisco, 1972.

Pilati, M. (1981). Imperial College preprints 81-32 and 81-37.

Ryan, R. M., and Shepley, L. C. (1975). *Homogeneous Relativistic Cosmologies*. Princeton University Press, Princeton, New Jersey.

Thirring, W. E. (1958). *Principles of Quantum Electrodynamics*. Academic Press, New York.